The Plactic Monoid

REPRESENTATION THEORY

$V = \mathbb{C}^n$ $\quad GL(n, \mathbb{C})$-MODULE.

$\otimes^k V$ "ESSENTIALLY" CONTAINS

ALL FINITE-DIM'L IRR. REPS.

CORRECTLY: IF V_λ IS AN IRR.

THEN THE HIGHEST WEIGHT

$\lambda = (\lambda_1, \dots, \lambda_n)$ $\quad \lambda_i \in \mathbb{Z}$

$\lambda_1 \geq \dots \geq \lambda_n$

IF $\lambda_n \geq 0$ THEN λ IS A PARTITION.

IF λ IS A PARTITION OF k THEN

V_λ OCCURS IN $\otimes^k V$.

THE λ_i MIGHT BE NEGATIVE

$$V \otimes \det^m = V_{(\lambda_1+m, \lambda_2+m, \dots, \lambda_n+m)}$$

BY CHOOSING $m \geq -\lambda_n$ THIS IS A PARTITION

$$\det^m \otimes V_\lambda \subseteq \otimes^k V$$

SO EVEN IF λ IS NOT A PARTITION,

$$V_\lambda \subseteq (\otimes^k V) \otimes \det^{-m}$$

UNIMPORTANT.

SO ESSENTIALLY ALL IRR. REPS ARE CONTAINED IN $\otimes^k \mathbb{C}^n$ FOR SUITABLE k.

λ A PARTITION NOW

$\lambda \vdash k \qquad k = \sum \lambda_i$

$$V_\lambda \subseteq \otimes^k V$$

PERHAPS SEVERAL TIMES.

$$\otimes^k V = \sum_{\lambda \vdash k} V_\lambda \otimes V_\lambda^{S_k}$$

IS $GL(n, \mathbb{C}) \times S_n$ MODULE.

S_n ACCOUNTS FOR MULTIPLE OCCURENCES.

COMBINATORICS:

RSK IS A COMBINATORIAL ANALOGUE. CRYSTAL LANGUAGE IS GOOD FOR UNDERSTANDING THIS BUT THE BASIC ALGORITHMS (ROBINSON, SCHENSTED, KNUTH, LASCOUX, SCHÜTZENBERGER) PREDATE CRYSTAL. LS INVENTED THE "PLACTIC MONOID" WHICH IS ALSO A GOOD WAY OF UNDERSTANDING THIS. THERE ARE CHAPTERS ABOUT PLACTIC MONOID IN BUMP-SCHILLING FULTON: YOUNG TABLEAUX, M. LOTHAIRE ALGEBRAIC COMBINATORICS ON WORDS.

ANALOG OF $\otimes^k V$

IS $\otimes^k \mathcal{B}$

$\mathcal{B}$ IS STANDARD CRYSTAL

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \rightarrow \cdots \xrightarrow{n-1} \boxed{n}$$

$\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_n}$ CAN BE

REP'D AS A WORD $a_1 a_2 \cdots a_k$.

$\otimes^k \mathcal{B}$ MAY CONTAIN MULTIPLE

COPIES OF $\mathcal{B}_\lambda$ ← CRYSTAL OF TABLEAUX

(ONE COPY FOR EACH STANDARD
TABLEAU OF SHAPE λ.)

WE CAN DEFINE AN EQUIVALENCE

RELATION ON WORDS

$a_1 \cdots a_n \mapsto \boxed{a_1} \otimes \cdots \otimes \boxed{a_n}$.

$a_i \in \{1, 2, \cdots, n\}$

LET $v, w \in \otimes^k \mathbb{B}$.

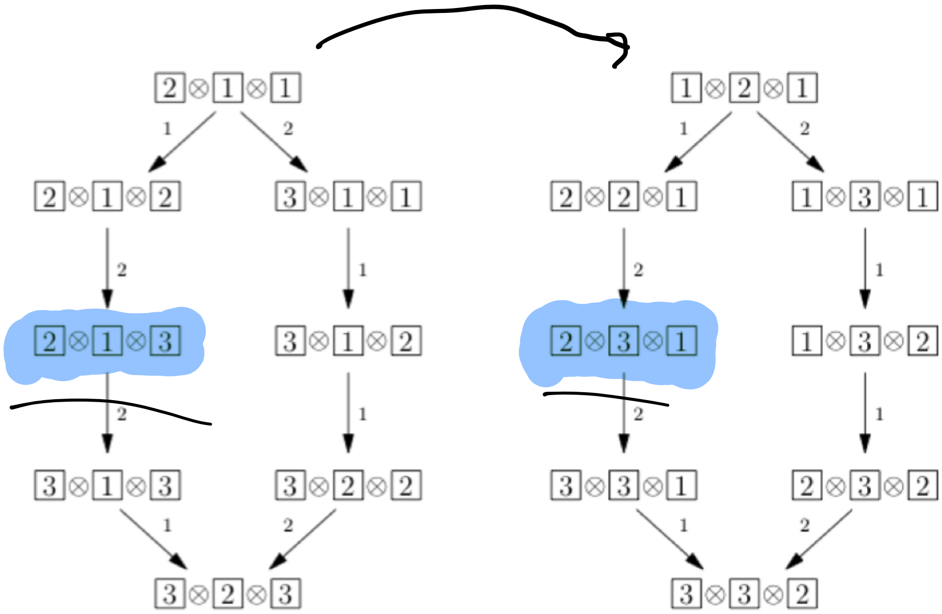
I WILL SAY v, w ARE PLACTICALLY EQUIVALENT IF THEY LIVE IN ISOMORPHIC CONNECTED SUBCRYSTALS.

C_v = CONNECTED COMPONENT OF $\otimes^k \mathbb{B}$ CONTAINING v.

$v \sim w$ MEANS $C_v \cong C_w$ AND THE ISOMORPHISM IS UNIQUE.

IT IS ASSUMED IT TAKES v TO w.

$\otimes^3 \mathbb{B}$ $\quad n = 3$



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OBVIOUSLY USEFUL SINCE IF I IDENTIFY ELTS THAT ARE PLACTICALLY EQUIVALENT

$$\bigcup_n \otimes^n \mathcal{B} \Big/ \text{PLACTIC EQUIVALENCE}$$

BECOMES THE DISJOINT UNION OF ALL CRYSTALS $\mathcal{B}_\lambda$ OF TABLEAUX.

$$\bigvee_n \otimes^k \mathcal{B}$$ CAN BE IDENTIFIED WITH SET OF WORDS IN ALPHABET $\{1, 2, \dots, n\}$

SO THIS QUOTIENT HAS A MULTIPLICATIVE STRUCTURE.

TENSOR PRODUCT OPERATION

$$\left(\otimes^k \mathcal{B}\right) \otimes \left(\otimes^\ell \mathcal{B}\right) \to \otimes^{k+\ell} \mathcal{B}$$

IS OBVIOUSLY COMPATIBLE WITH PLACTIC EQUIVALENCE.

$v, w \in \otimes^k \mathcal{B}$. $\quad v \sim w$

$x, y \in \otimes^\ell \mathcal{B}$ $\quad x \sim y$

$$v \otimes x \sim w \otimes y$$

BOTH LIE IN ISOMORPHIC SUBCRYSTALS

$$v \otimes x \in \mathcal{C}_v \otimes \mathcal{C}_x \cong \mathcal{C}_w \otimes \mathcal{C}_y .$$

PLACTIC MONOID.

GIVEN $v \in \otimes^k \mathcal{B}$

$\mathcal{C}_v$ CONNECTED COMPONENT $\ni v$

$\mathcal{C}_v \cong \mathcal{B}_\lambda \qquad \lambda \vdash k.$

$\dim \Pi_v^{S_\lambda}$ = # OF STANDARD TABLEAUX OF SHAPE λ

= # OF WORDS PLACTICALLY EQUIVALENT TO v.

PLACTIC EQUIVALENCE =
KNUTH EQUIVALENCE OF WORDS.

OPERATION ON WORDS

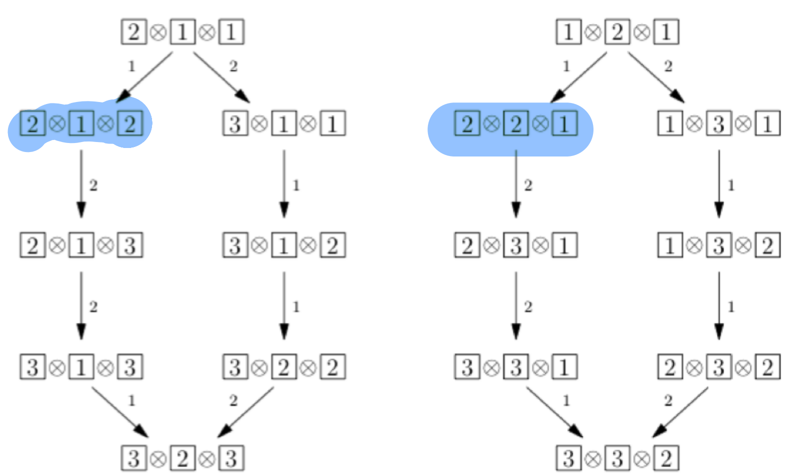
$a_1 \cdots a_k$

CHANGES ELEMENTS TWO AT A TIME LOOKING AT TRIPLES.

<u>DEF.</u> IF $a < b \leq c$ $\qquad bac \sim bca$

IF $a \leq b < c$ $\qquad acb \sim cab$

GIVEN A LONGER WORD ALLOW ANY TRIPLE TO BE CHANGED THIS WAY



$212 \sim 221$

USE FIRST CRITERION

$bca = 221 \qquad a < b \leq c$

$\qquad\qquad\qquad\;\; 1 \quad 2 \quad 2$

$\sim bac \qquad 212$

THEOREM: KNUTH EQUIV. = PLACTIC EQUIV.

RSK AND CRYSTALS.

COMES FROM

$$\to \quad \mathcal{B}_{(k)} \otimes \mathcal{B} = \mathcal{B}_{(k+1)} \sqcup \mathcal{B}_{(k,1)}$$

$\uparrow$

CRYSTAL OF ROWS

$$\boxed{a_1 \cdots a_n} = \boxed{a_1} \otimes \boxed{a_2} \otimes \cdots$$

$a_1 \leq a_2 \leq \cdots$

$$\boxed{a_1 \cdots a_n} \otimes \boxed{c}$$

IN ISOMORPHISM $\mathcal{B}_{(k)} \quad \mathcal{B}$

$$B_{(n+1)} \to \boxed{a_1 \cdots a_n c} \qquad \text{IF} \quad c \geq a_n$$

$$B_{(n,1)} \quad \begin{array}{|c|}\hline a_1 \cdots c \cdots a_n \\ \hline a_k \\ \hline \end{array} \qquad \text{DO SCHENSTED INSERTION}$$

THM ANY ELT OF $\otimes^n B$ IS PLACTICALLY EQUIVALENT TO AN ELT OF B_λ FOR SOME $\lambda \vdash n$.

PROOF. BY REPEATEDLY DOING SCHENSTED INSERTION

$$\underbrace{\boxed{a_1} \cdots \boxed{a_n}} \otimes \boxed{c} \sim$$

$$T \left\{ \begin{array}{|l|} \hline R_1 \\ R_2 \\ \vdots \\ R_m \\ \hline \end{array} \right. \otimes \boxed{c}.$$

INDUCTION HYPOTHESIS: ANY ELT OF $\otimes^n B$ IS PLACTICALLY EQUIV TO

$\mathcal{B}_\lambda$ for some λ.

$$v = \underbrace{R_m \otimes \cdots \otimes}_{\substack{\text{one realization} \\ \text{of } \mathcal{B}_\lambda \text{ in } \otimes^n \mathcal{B}}} R_1$$

$$v \otimes \boxed{c} = R_m \otimes \cdots \otimes R_1 \otimes \boxed{c}$$

$$R_1 \otimes \boxed{c} \in \mathcal{B}_{(\lambda_1)} \otimes \mathcal{B}$$

$\sim$ either elt of $\mathcal{B}_{(\lambda_1+1)}$ or $\mathcal{B}_{(\lambda_1, 1)}$

In second case

$$R_1 \otimes \boxed{c} \sim \begin{array}{|l|}\hline \quad R' \quad \\ \hline \end{array} \atop {\begin{array}{|c|}\hline c' \\ \hline\end{array}} = \boxed{c'} \otimes R_1'$$

$$R_m \otimes \cdots \otimes R_1 \otimes \boxed{c} \sim$$

$$R_m \otimes \cdots R_2 \otimes \boxed{c'} \otimes R_1'.$$

REPEAT UNTIL END UP IN

$\mathcal{R}_m \in \mathcal{B}_{(\lambda_m)}$ $\quad \mathcal{R}_m \otimes \boxed{\lambda}$ $\quad \mathcal{B}_{(\lambda_m+1)}$ $\quad \mathcal{B}_{(\lambda_m, 1)}$

$$\mathcal{R}_m \otimes \cdots \mathcal{R}_{m+1} \otimes \mathcal{R}_m' \otimes \cdots$$

ALGORITHM IS JUST RSK.

$1 \; 2 \; 2 \; 1 \sim 2 \; 1 \; 1 \; 2.$

$$\boxed{1}\,\boxed{2}\,\boxed{2} \otimes \boxed{1}$$

$$\begin{array}{ccc} \boxed{1} & \boxed{1} & \boxed{2} \\ \boxed{2} & & \end{array} \qquad 2 \; 1 \; 1 \; 2 \in \mathcal{B}_{(3,1)}.$$

DEF. IF $a < b \leq c$ $\quad bac \sim bca$

IF $a \leq b < c$ $\quad acb \sim cab$

WHY DOES KNUTH EQUIV. $\Rightarrow$ PLACTIC EQUIVALENCE ?

PLACTIC MONOID IS ASSOCIATIVE.

$$a_1 \cdots a_m \, (bca) \, a_{m+1} \cdots a_n .$$

USE RSK ON (bca) TO OBTAIN A PLACTICALLY EQUIV. WORD.

FIRST CASE $a < b \leq c$

$$b\,c\,a \sim \boxed{b\ c}\ a\ a \qquad \begin{array}{ll} a & c \\ b & \end{array}$$

$$= bac$$

$bca \sim bac$.

$$\bigotimes^m \mathcal{B} \otimes \left(\bigotimes^3 \mathcal{B} \right) \otimes \left(\bigotimes^{n-m-3} \mathcal{B} \right)$$

$\mathcal{B}_\infty$ CRYSTAL.